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Angular symmetry breaking induced by electromagnetic field

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Abstract. It is well known that velocities do not commute in the presence of an electromagnetic field. This property implies that angular algebra symmetries, such as the sO(3) and Lorentz algebra symmetries, are broken. To restore these angular symmetries we show the necessity of adding the Poincaré momentum **M** to the simple angular momentum **L**. These restorations performed successively in a flat space and in a curved space lead in each case to the generation of a Dirac magnetic monopole. In the particular case of the Lorentz algebra we consider an application of our theory to gravitoelectromagnetism. In this last case we establish a qualitative relation giving the mass spectrum for dyons.

1 Introduction

The concept of symmetry breaking is fundamental in science and particularly in physics. In this work we will focus our attention on the effects of the non-commutativity on the angular algebra symmetries such as the sO(3) algebra symmetry and the Lorentz algebra symmetry. Theories in non-commutative geometry have been at the center of recent interest [1,2]. Instead of the non-commutativity between coordinates $\mathbf{x} = \left\{x^i\right\}_{i=1,N}$ here we will rather use non-commutativity between the velocities $\dot{\mathbf{x}} = \left\{\dot{x}^i\right\}_{i=1,N}$, leaving the study of the more general case where neither coordinates nor velocities commute for another paper [3]. As we will see in the following, in the tangent bundle space $T(\mathcal{M})$ of a manifold $\mathcal{M}$ endowed with a Poisson structure, the non-commutativity between velocities implies a gauge curvature $F^{ij}(\mathbf{x})$, i.e. an electromagnetic field. This gauge curvature breaks explicitly the angular algebra symmetry of the dynamical system. The particular case of the breaking of the Lorentz algebra in the presence of a covariant Hamiltonian was already investigated in a recent paper [4]. Here we embedded the formalism used in [4] in a more general one in, defining a Poisson structure where the dynamics are governed by a covariant Hamiltonian. We also show that the brackets defining the Poisson structure on $T(\mathcal{M})$ are equivalent (at least up to the second order) to Moyal brackets defined on the tangent bundle space $T(\mathcal{M})$ where the non-commutative parameters are related to the electromagnetic field.

Contrary to the standard approach used commonly in the study of the gauge theories, we do not settle our formalism in the cotangent bundle space $T^*(\mathcal{M})$, i.e. the space of coordinates $\{\mathbf{x}, \mathbf{p}\}$, but we continue the prospection of the gauge theories in the tangent bundle space $T(\mathcal{M})$, i.e. the space of coordinates $\{\mathbf{x}, \dot{\mathbf{x}}\}$. The main reason for this choice is that the generalized momentum $\mathbf{p}$ is not gauge invariant, and this is particularly important when we consider angular algebra symmetry. Indeed, the Lie algebra is always trivially realized when the angular momentum $\mathbf{L}$ is expressed in terms of the generalized momentum $\mathbf{p}$. In the tangent bundle space $T(\mathcal{M})$ we show that the angular algebra symmetry is broken by the electromagnetic field issuing from the non-commutativity between velocities. This angular algebra symmetry is restored by the introduction of the Poincaré momentum [5] which implies the existence of a Dirac magnetic monopole [6–8]. Here we insist more than in [7] on the link between the restoration of the angular Lie algebra symmetries and the generation of magnetic monopoles. Note that a similar monopole also arises from the electromagnetic U(1) gauge theory when one requires the dual symmetry under rotation of the electric and magnetic fields in the free Maxwell equations. In our case we do not need the dual symmetry (except in Sect. 4 where we use the Hodge duality for other purposes) since the monopole field naturally arises from the restoration of the angular algebra symmetry which was broken by the electromagnetic field. In this paper we perform the restoration for the sO(3) algebra symmetry successively in a flat and in a curved space and then for the local Lorentz algebra symmetry. Finally, at the end of the paper, using the Hodge duality on our theory and the recent work of Mashhoon [9] we develop the application to the gravitoelectromagnetism theory.

We would like to remark that the Moyal brackets we use in this paper are closely connected to those introduced by Feynman in his remarkable demonstration of the Maxwell

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equations where he tried to develop a quantization procedure without the resort to a Lagrangian or a Hamiltonian. Feynman's ideas were exposed by Dyson in an elegant publication [10]. The interpretation of Feynman's derivation of the Maxwell equations has aroused great interest among physicists. In particular, Tanimura [11] has generalized Feynman's derivation in a Lorentz covariant form with a scalar time evolution parameter. An extension of Tanimura's approach has been achieved [12] in using the Hodge duality to derive the two groups of Maxwell equations with a magnetic monopole in flat and curved spaces. In [13] the descriptions of relativistic and non-relativistic particles in an electromagnetic field was studied, whereas in [14] a dynamical equation for spinning particles was proposed. A rigorous mathematical interpretation of Feynman's derivation connected to the inverse problem for Poisson dynamics has been formulated in [15]. Also the papers of [16,17] considered Feynman's derivation in the frame of Helmholtz's inverse problem for the calculus of variations. More recently, some works [18–20] have provided new looks on the Feynman's derivation of the Maxwell equations. One may mention a tentative extension of Feynman's derivation of the Maxwell equations to the case of non-commutative geometry using the standard Moyal brackets [21].

This paper is organized as follows. In Sect. 2 we introduce the formalism used throughout the paper. The dynamical equation will be given by a covariant Hamiltonian defined as in Goldstein's textbook [22] and by deformed Poisson brackets. We show that these deformed Poisson brackets can be defined as the second order approximation of generalized Moyal brackets. In Sect. 3 we recall how to restore the sO(3) algebra symmetry in the case of a flat space [4] and we extend the restoration to the case of a curved space. In Sect. 4 we restore the Lorentz algebra symmetry in a curved space and we propose an application of our formalism to the case of the gravitoelectromagnetism theory. We then derive a qualitative relation giving the mass spectrum for dyons. In Sect. 5 we summarize the main achievements of this work.

2 Mathematical foundations

Let $\mathcal{M}$ be a N-dimensional vectorial manifold diffeomorphic to $\mathbb{R}^N$. Let a particle with a mass m and an electrical charge q be described by the vector $\mathbf{x} = \{x^i\}_{i=1,\dots,N}$ which defines its position on the manifold $\mathcal{M}$. Let τ be the parameter of the group of diffeomorphisms $\mathcal{G} : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ such as $\mathcal{G}(\tau, \mathbf{x}) = \mathcal{G}^\tau \mathbf{x} = \mathbf{x}(\tau)$. Then taking τ as the time parameter of our physical system we are able to define a velocity vector $\dot{\mathbf{x}} \in \mathcal{M}$ as $\dot{\mathbf{x}} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} = \mathcal{G}^\tau \mathbf{x} = \{\dot{x}^i(\tau)\}_{i=1,\dots,N}$. Let $T(\mathcal{M})$ be the tangent bundle space associated with the manifold $\mathcal{M}$; a point on $T(\mathcal{M})$ is then described by a $2N$-dimensional vector $\mathbf{X} = \{\mathbf{x}, \dot{\mathbf{x}}\}$.

2.1 Poisson structure

Let $A^0(T(\mathcal{M})) = C^\infty(T(\mathcal{M}), \mathbb{R})$ be the algebra of differential functions defined on the manifold $T(\mathcal{M})$. We define a Poisson structure on $T(\mathcal{M})$ which is an internal skew-symmetric bilinear multiplicative law on $A^0(T(\mathcal{M}))$ denoted $(f, g) \to [f, g]$ and satisfying the Leibniz rule

$$[f, gh] = [f, g]h + [f, h]g \tag{1}$$

and the Jacobi identity

$$J(f, g, h) = [f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0. \tag{2}$$

The manifold $T(\mathcal{M})$ with such a Poisson structure is called a Poisson manifold. We define a dynamical system on the Poisson manifold $T(\mathcal{M})$ by the following differential equation:

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = [f, H], \tag{3}$$

where $H \in A^0(T(\mathcal{M}))$ is the Hamiltonian of the dynamical system.

With such definitions we derive the following important relations for functions belonging to $A^0(T(\mathcal{M}))$:

$$\begin{aligned}[f(\mathbf{X}), h(\mathbf{X})] &= \{f(\mathbf{X}), h(\mathbf{X})\} \\ &+ \left[x^i, x^j\right] \frac{\partial f(\mathbf{X})}{\partial x^i} \frac{\partial h(\mathbf{X})}{\partial x^j} + \left[\dot{x}^i, \dot{x}^j\right] \frac{\partial f(\mathbf{X})}{\partial \dot{x}^i} \frac{\partial h(\mathbf{X})}{\partial \dot{x}^j},\end{aligned} \tag{4}$$

where we have introduced the Poisson-like brackets defined by

$$\begin{aligned}&\{f(\mathbf{X}), g(\mathbf{X})\} \\ &= \left[x^i, \dot{x}^j\right] \left(\frac{\partial f(\mathbf{X})}{\partial x^i} \frac{\partial h(\mathbf{X})}{\partial \dot{x}^j} - \frac{\partial f(\mathbf{X})}{\partial \dot{x}^i} \frac{\partial h(\mathbf{X})}{\partial x^j} \right).\end{aligned} \tag{5}$$

We can think of the relation (4) as the simple deformation of the Poisson-like brackets introduced in (5). It is obvious that the tensors $\left[x^i, x^j\right]$ and $\left[\dot{x}^i, \dot{x}^j\right]$ are skew symmetric. We introduce then the following notation:

$$\left[x^i, x^j\right] = \lambda \Theta^{ij}(\mathbf{X}) \ , \ \lambda \in \mathbb{R}, \tag{6}$$

$$\left[x^i, \dot{x}^j\right] = \gamma G^{ij}(\mathbf{X}) \ , \ \gamma \in \mathbb{R}, \tag{7}$$

$$\left[\dot{x}^i, \dot{x}^j\right] = \gamma' \mathcal{F}^{ij}(\mathbf{X}) \ , \ \gamma' \in \mathbb{R}, \tag{8}$$

where $G^{ij}(\mathbf{X})$ is a priori any $N \times N$ tensor, and where $\Theta^{ij}(\mathbf{X})$ and $\mathcal{F}^{ij}(\mathbf{X})$ are two $N \times N$ skew-symmetric tensors, $\mathcal{F}^{ij}(\mathbf{X})$ being related to the electromagnetic tensor introduced in a preceding paper [8] .

In the following we will require the property of locality,

$$\Theta^{ij}(\mathbf{X}) = 0, \tag{9}$$

leaving the study of the $\Theta^{ij}(\mathbf{X}) \neq 0$ case for another work [3]. The property (9) expresses the commutativity of the internal skew-symmetric bilinear law involving positions $\mathbf{x}$ on the manifold $\mathcal{M}$ whereas taking $\mathcal{F}^{ij}(\mathbf{X}) \neq 0$ implies non-commutativity between the velocities $\dot{\mathbf{x}}$.

The property of locality (9) and the fact that the velocity vector has to verify the dynamical equation (3) imply the following general expression for the Hamiltonian:

$$H = \frac{1}{2} m g_{ij}(\mathbf{x}) \dot{x}^i \dot{x}^j + f(\mathbf{x}), \tag{10}$$

where we take $\lambda = 1/m$ and where

$$G^{ij}(\mathbf{X}) = g^{ij}(\mathbf{x}) \tag{11}$$

is now the metric tensor defined on the manifold $\mathcal{M}$, the function $f(\mathbf{x})$ being any only position dependent function belonging to $A^0(T(\mathcal{M}))$.

2.2 Generalized Moyal brackets

In this section we show that we can embed our construction in a more general formalism by introducing a generalization of the Moyal brackets defined over the tangent bundle space. Indeed we will show that the brackets defined by the relation (4) can be considered as the second order expansion of the Moyal brackets where the roles of the non-commutative parameters are played by the tensors Θ, $\mathcal{F}$ and G defined in the preceding section.

Let now f and h be functions belonging to $A^0(T(\mathcal{M}))$. Then we can define a Moyal star product $\star : T(\mathcal{M}) \times T(\mathcal{M}) \to T(\mathcal{M})$ such as

$$f(\mathbf{X}) \star h(\mathbf{X}) = f(\mathbf{X}) \exp\left(A\left(\mathbf{X},\mathbf{Y}\right)\right) h(\mathbf{Y})|_{\mathbf{X}=\mathbf{Y}}\,, \tag{12}$$

where

$$A\left(\mathbf{X},\mathbf{Y}\right) = \frac{1}{2}\overleftarrow{\frac{\partial}{\partial \mathbf{X}^\alpha}} a^{\alpha\beta}(\mathbf{X})\overrightarrow{\frac{\partial}{\partial \mathbf{Y}^\beta}}, \quad \alpha,\beta = 1,\ldots,2N. \tag{13}$$

Here $\mathbf{X},\mathbf{Y} \in T(\mathcal{M})$, and the differential operators

$$\overleftarrow{\frac{\partial}{\partial \mathbf{X}^\alpha}}, \quad \overrightarrow{\frac{\partial}{\partial \mathbf{Y}^\beta}}$$

are understood to act respectively on the left and on the right side of the expression. To this non-commutative product $\star$ we can associate a particular commutator which is known as the Moyal brackets,

$$[f(\mathbf{X}), h(\mathbf{X})]_\star = f(\mathbf{X}) \star h(\mathbf{X}) - h(\mathbf{X}) \star f(\mathbf{X}). \tag{14}$$

Now if we give to the $2N \times 2N$ tensor $a^{\alpha\beta}$ in (13) the following antisymmetric form:

$$a^{\alpha\beta}(\mathbf{X}) = \begin{pmatrix} \gamma\,\Theta(\mathbf{X}) & -\lambda G(\mathbf{X}) \\ \lambda G(\mathbf{X}) & \gamma'\mathcal{F}(\mathbf{X}) \end{pmatrix}, \quad \lambda,\gamma,\gamma' \in \mathbb{R}, \tag{15}$$

we can show that the Moyal brackets (14) are, similarly to the brackets defined in the preceding section, also a simple deformation of the Poisson-like brackets introduced in (5). Indeed if we develop the Moyal brackets up to the second order in λ, γ and γ' we obtain

$$[f(\mathbf{X}), h(\mathbf{X})]_\star = \{f(\mathbf{X}), h(\mathbf{X})\} \tag{16}$$
$$+\gamma\,\Theta^{ij}(\mathbf{X})\frac{\partial f(\mathbf{X})}{\partial x^i}\frac{\partial h(\mathbf{X})}{\partial x^j} + \gamma'\mathcal{F}^{ij}(\mathbf{X})\frac{\partial f(\mathbf{X})}{\partial \dot{x}^i}\frac{\partial h(\mathbf{X})}{\partial \dot{x}^j}.$$

We recall that we will limit ourselves to the case $\Theta^{ij}(\mathbf{X}) = 0$ and will omit the $\star$ symbol in the following.

3 sO(3) algebra

In this section we will particularly focus on the consequences of the breaking of the sO(3) symmetry in a flat space and in a curved space as well. In three dimensional space the derivation of the Maxwell equations is quite formal and we will consider only the case of magnetostatic and electrostatic fields. The time dependent fields case can easily be derived by adding an explicit time derivative to the dynamical law (3). We do not have to add this term for the four dimensional space case where the Maxwell equations are intrinsically Lorentz covariant, since the fields are time dependent by construction.

3.1 sO(3) algebra in flat space

In a three dimensional flat space we have $g^{ij}(\mathbf{x}) = \delta^{ij}$ and the Hamiltonian (10) of the Poisson structure then reads

$$H = \frac{1}{2}m\dot{x}^i\dot{x}_i + f(\mathbf{x}). \tag{17}$$

Before considering the sO(3) algebra, let us first derive the particle equation of motion and the Maxwell field equations from our formalism.

3.1.1 Maxwell equations

The Jacobi identity (2) involving position and velocity components,

$$\frac{m}{\gamma'}J(x^i,\dot{x}^j,\dot{x}^k) = \frac{\partial\mathcal{F}^{jk}(\mathbf{X})}{\partial \dot{x}^i} = 0, \tag{18}$$

shows that the gauge curvature is velocity independent, $\mathcal{F}^{ij}(\mathbf{X}) \equiv F^{ij}(\mathbf{x})$. From the Jacobi identity (2) involving only velocities' components we derive the Bianchi equation,

$$\frac{m}{\gamma'}J(\dot{x}^i,\dot{x}^j,\dot{x}^k) = \varepsilon^k{}_{ji}\frac{\partial F^{ij}(\mathbf{x})}{\partial x^k} = 0, \tag{19}$$

which, if we set $F^{ij}(\mathbf{x}) = \varepsilon^{ji}{}_k B^k(\mathbf{x})$, gives the following Maxwell equation:

$$\nabla\cdot\mathbf{B} = 0. \tag{20}$$

Now using the dynamical equation (3) we obtain the following equation of motion:

$$m\ddot{x}^i = m\left[\dot{x}^i, H\right] = qF^{ij}(\mathbf{x})\dot{x}_j + qE^i\left(\mathbf{x}\right), \tag{21}$$

where

$$qE^i\left(\mathbf{x}\right) = -\frac{\partial f(\mathbf{x})}{\partial x_i}. \tag{22}$$

We then have a particle of mass m and electrical charge q moving in flat space in the presence of a magnetostatic and an electrostatic external field. In order to get the usual form (21) for the equation of motion we have set $\gamma' = q/m^2$ in the definition (8).

We are able now to derive the other Maxwell equation of the first group. With the dynamical equation (3) we express the time derivative of the magnetic field,

$$\frac{\mathrm{d}B^i}{\mathrm{d}t} = \frac{1}{2}\varepsilon^i{}_{jk}\left[F^{jk}, H\right] = \frac{1}{2\gamma'}\varepsilon^i{}_{jk}\left[\left[\dot{x}^j, \dot{x}^k\right], H\right], \qquad (23)$$

and we use the Jacobi identy (2) to rewrite the last term of the last equation. After some calculus we obtain

$$\frac{\mathrm{d}B^i}{\mathrm{d}t} = -\dot{x}^i \nabla\cdot\mathbf{B} + \frac{\partial B^i}{\partial x_j}\dot{x}_j + \varepsilon^i{}_{jk}\frac{\partial E^j}{\partial x_k}, \qquad (24)$$

which using (20) gives the second Maxwell equation

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} = \mathbf{0} \qquad (25)$$

for static fields and electric fields deriving from any potential $f(\mathbf{x})$ (22).

As the two other Maxwell equations are not Galilean covariant they cannot be deduced from the formalism and can be merely seen as a definition of the charge density and the current density. Nevertheless, as shown in the next section the complete set of Maxwell equations can be deduced in the relativistic generalization.

3.1.2 sO(3) algebra and Poincaré momentum

One of the most important symmetries in physics is naturally the spherical symmetry corresponding to the isotropy of the physical space. This symmetry is related to the sO(3) algebra. In the following we show that this symmetry is broken when an electromagnetic field is applied. In order to study the symmetry breaking of the sO(3) algebra we use the usual angular momentum $L^i = m\varepsilon^i{}_{jk}x^j\dot{x}^k$ which is a constant of motion in the absence of a gauge field. In fact, there being no electromagnetic field implies $F^{ij}(\mathbf{x}) = \left[\dot{x}^i, \dot{x}^j\right] = 0$, and the expression of the sO(3) Lie algebra with our brackets (4) gives then the standard algebra defined in terms of the Poisson brackets (5)

$$\begin{cases} \left[x^i, L^j\right] = \left\{x^i, L^j\right\} = \varepsilon^{ij}{}_k x^k, \\ \left[\dot{x}^i, L^j\right] = \left\{\dot{x}^i, L^j\right\} = \varepsilon^{ij}{}_k \dot{x}^k, \\ \left[L^i, L^j\right] = \left\{L^i, L^j\right\} = \varepsilon^{ij}{}_k L^k. \end{cases} \qquad (26)$$

When the electromagnetic field is turned on, this algebra is broken in the following manner:

$$\begin{cases} \left[x^i, L^j\right] = \left\{x^i, L^j\right\} = \varepsilon^{ij}{}_k x^k, \\ \left[\dot{x}^i, L^j\right] = \left\{\dot{x}^i, L^j\right\} + \frac{q}{m}\varepsilon^j{}_{kl}x^k F^{il}(\mathbf{x}) \\ \qquad = \varepsilon^{ij}{}_k \dot{x}^k + \frac{q}{m}\varepsilon^j{}_{kl}x^k F^{il}(\mathbf{x}), \\ \left[L^i, L^j\right] = \left\{L^i, L^j\right\} + q\varepsilon^i{}_{kl}\varepsilon^j{}_{ms}x^k x^m F^{ls}(\mathbf{x}) \\ \qquad = \varepsilon^{ij}{}_k L^k + q\varepsilon^i{}_{kl}\varepsilon^j{}_{ms}x^k x^m F^{ls}(\mathbf{x}). \end{cases} \qquad (27)$$

In order to restore the sO(3) algebra we introduce a new angular momentum $M^i(\mathbf{X})$ which is a priori position and velocity dependent. We consider then the following transformation law:

$$L^i(\mathbf{X}) \to \mathcal{L}^i(\mathbf{X}) = L^i(\mathbf{X}) + M^i(\mathbf{X}), \qquad (28)$$

and we require that this new angular momentum $\mathcal{L}^i$ verifies the usual sO(3) algebra,

$$\begin{cases} \left[x^i, \mathcal{L}^j\right] = \left\{x^i, \mathcal{L}^j\right\} = \varepsilon^{ij}{}_k x^k, \\ \left[\dot{x}^i, \mathcal{L}^j\right] = \left\{\dot{x}^i, \mathcal{L}^j\right\} = \varepsilon^{ij}{}_k \dot{x}^k, \\ \left[\mathcal{L}^i, \mathcal{L}^j\right] = \left\{\mathcal{L}^i, \mathcal{L}^j\right\} = \varepsilon^{ij}{}_k \mathcal{L}^k. \end{cases} \qquad (29)$$

These equations, (29), then give three constraints on the expression of the angular momentum $\mathcal{L}^i$. From the first relation in (29) we easily deduce that M^i is velocity independent,

$$M^i(\mathbf{X}) = M^i(\mathbf{x}), \qquad (30)$$

and from the second relation we obtain

$$\left[\dot{x}^i, M^j\right] = -\frac{1}{m}\frac{\partial M^j(\mathbf{x})}{\partial x_i} = -\frac{q}{m}\varepsilon^j{}_{kl}x^k F^{il}(\mathbf{x}), \qquad (31)$$

and finally the third relation gives

$$M^i = \frac{1}{2}q\varepsilon_{jkl}x^i x^k F^{jl}(\mathbf{x}) = -q\left(\mathbf{x}\cdot\mathbf{B}\right)x^i. \qquad (32)$$

Equations (31) and (32) are compatible only if the magnetic field $\mathbf{B}$ is the Dirac magnetic monopole field,

$$\mathbf{B} = \frac{g}{4\pi}\frac{\mathbf{x}}{\|\mathbf{x}\|^3}. \qquad (33)$$

The vector $\mathbf{M}$ allowing us to restore the sO(3) symmetry (29) is then the Poincaré momentum [5]

$$\mathbf{M} = -\frac{qg}{4\pi}\frac{\mathbf{x}}{\|\mathbf{x}\|},$$

already found in a preceding paper [4]. The total angular momentum is then

$$\mathcal{L} = \mathbf{L} - \frac{qg}{4\pi}\frac{\mathbf{x}}{\|\mathbf{x}\|}. \qquad (34)$$

This expression was initially found by Poincaré in a different context [5]. Actually he was looking for a new angular momentum that would be a constant of motion. In our framework this property is trivially verified by using the dynamical relation (3).

Let us now discuss an important point. As the Dirac magnetic monopole is located at the origin we have

$$J(\dot{x}^i, \dot{x}^j, \dot{x}^k) = \nabla\cdot\mathbf{B} = g\delta^3(\mathbf{x}). \qquad (35)$$

The preservation of the sO(3) symmetry in the presence of a gauge field is then incompatible with the requirement of the Jacobi identity at the origin of the coordinates and we have to exclude the origin from the manifold $\mathcal{M}$. As the Jacobi identity is the infinitesimal statement of associativity in the composition law of the translation group [23], the breakdown of the Jacobi identity (35) when $\nabla\cdot\mathbf{B} \neq 0$

implies that finite translations do not associate. In the usual quantum mechanics non-associativity between operators acting on the Hilbert space cannot be tolerated; one has to use Dirac's quantization procedure to save the associativity (35).

In order to consider quantum mechanics within our framework we have to quantify as usual the total angular momentum $\mathcal{L}$. Considering the rest frame of the particle we have the following Dirac quantization:

$$\frac{gq}{4\pi} = \frac{n}{2}\hbar. \tag{36}$$

Note that the Poincaré momentum is also related to the Wess–Zumino term introduced by Witten in the case of a simple mechanical problem [24]. Indeed let us consider a particle of mass m constrained to move on a two dimensional sphere of radius 1 with a spatial-temporal reflection symmetry. The system can be seen as a particle submitted to a strength having the following form: $qg\varepsilon_{ijk}x^k\dot{x}^j$, which is interpreted by Witten as a Lorentz force acting on an electric charge q in interaction with a magnetic monopole of magnetic charge g located at the center of the sphere. In the quantum version of this system Witten has recovered the Dirac quantization condition by means of topological techniques.

3.2 sO(3) algebra in curved space

The Hamiltonian is now defined in a curved space by

$$H = \frac{1}{2}mg_{ij}(\mathbf{x})\dot{x}^i\dot{x}^j + f(\mathbf{x}), \tag{37}$$

where the metric $g_{ij}(\mathbf{x})$ is now position dependent.

3.2.1 Maxwell equations

The commutation relations for contravariant components are now in a curved space

$$\begin{cases} \left[x^i, x^j\right] = 0, \\ \left[x^i, \dot{x}^j\right] = \dfrac{1}{m}g^{ij}(\mathbf{x}), \\ \left[\dot{x}^i, \dot{x}^j\right] = \dfrac{q}{m^2}\mathcal{F}^{ij}(\mathbf{X}). \end{cases} \tag{38}$$

Then for covariant components we have

$$\begin{cases} [x_i, x_j] = 0, \\ [x_i, \dot{x}_j] = \dfrac{1}{m}g_{ij}(\mathbf{x}) + \dfrac{1}{m}\left(\partial_j g_{ik}\right)x^k, \\ [\dot{x}_i, \dot{x}_j] = \dfrac{q}{m^2}\mathcal{F}_{ij}(\mathbf{X}) + \dfrac{1}{m}\left(\partial_j g_{ik} - \partial_i g_{jk}\right)\dot{x}^k. \end{cases} \tag{39}$$

Using the Jacobi identity $J(x^k, \dot{x}^i, \dot{x}^j) = 0$ and using the fact that $g^{ik}g_{kj} = \delta^i{}_j$ we find the following expression for the general gauge field:

$$\frac{q}{m^2}\mathcal{F}_{ij}\left(\mathbf{X}\right) = g_{ik}g_{jl}\mathcal{F}^{kl}\left(\mathbf{X}\right) \tag{40}$$

$$= \frac{1}{m}\left(\partial_i g_{jk} - \partial_j g_{ik}\right)\dot{x}^k + \frac{q}{m^2}F_{ij}\left(\mathbf{x}\right),$$

where $F_{ij}(\mathbf{x})$ is a velocity independent gauge field. From the last equation in (39) we easily deduce that the commutator between covariant velocities is only position dependent,

$$[\dot{x}_i, \dot{x}_j] = \frac{q}{m^2}F_{ij}\left(\mathbf{x}\right). \tag{41}$$

Again with the dynamical equation (3) and with the above commutation relations we derive the equation of motion of a particle in a curved space in the presence of magnetostatic and electrostatic fields,

$$m\ddot{x}^i = -\Gamma^{i,jk}\dot{x}_j\dot{x}_k + qF^{ij}(\mathbf{x})\dot{x}_j + qE^i\left(\mathbf{x}\right), \tag{42}$$

where the Christoffel symbols are defined by

$$\Gamma^{i,jk} = \frac{1}{2}(-\partial^j g^{ik} + \partial^i g^{jk} - \partial^k g^{ij}). \tag{43}$$

The Jacobi identity $J\left(\dot{x}_i, \dot{x}_j, \dot{x}_k\right) = 0$ gives directly the first Maxwell equation of the first group,

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0, \tag{44}$$

and if we follow the same procedure as in the flat space we also recover the second Maxwell equation of the first group for the static fields,

$$\partial_t B_i = -\varepsilon_i{}^{jk}\partial_j E_k = 0.$$

As in the flat space case the second group of Maxwell equations are considered as the definition of the charge and the current density.

3.2.2 sO(3) algebra and Poincaré momentum

We define the angular momentum in a three dimensional curved space by the usual relations:

$$\begin{cases} L_i = m\sqrt{g(\mathbf{x})}\varepsilon_{ijk}x^j\dot{x}^k = m\mathrm{E}_{ijk}(\mathbf{x})x^j\dot{x}^k, \\ L^i = m\sqrt{g(\mathbf{x})}g^{ij}(\mathbf{x})\varepsilon_{jkl}x^k\dot{x}^l = m\mathrm{E}^{ijk}(\mathbf{x})x_j\dot{x}_k, \end{cases} \tag{45}$$

where $g(\mathbf{x}) = \det\left(g_{ij}(\mathbf{x})\right) = \left(\det\left(g^{ij}(\mathbf{x})\right)\right)^{-1}$. Using the relation $\partial_i g(\mathbf{x}) = g(\mathbf{x})g_{jk}(\mathbf{x})\partial_i g^{jk}(\mathbf{x})$ we easily show that the sO(3) algebra symmetry is broken in the following manner:

$$\begin{cases} \left[x^i, L^j\right] = \left\{x^i, L^j\right\} = \mathrm{E}^{ij}{}_k(\mathbf{x})x^k, \\ \left[\dot{x}^i, L^j\right] = \left\{\dot{x}^i, L^j\right\} + \dfrac{q}{m}\mathrm{E}^j{}_{kl}(\mathbf{x})x^k\mathcal{F}^{il}(\mathbf{X}) \\ \qquad = \mathrm{E}^{ij}{}_k(\mathbf{x})x^k - \dfrac{1}{2}\mathrm{E}^j{}_{kl}(\mathbf{x})x^k\dot{x}^l g_{mn}(\mathbf{x})\partial^i g^{mn}(\mathbf{x}) \\ \qquad + \dfrac{q}{m}\mathrm{E}^j{}_{kl}(\mathbf{x})x^k\mathcal{F}^{il}(\mathbf{X}), \\ \left[L^i, L^j\right] = \left\{L^i, L^j\right\} + q\mathrm{E}^i{}_{kl}(\mathbf{x})\mathrm{E}^j{}_{mn}(\mathbf{x})x^k x^m\mathcal{F}^{ln}\left(\mathbf{X}\right) \\ \qquad = \mathrm{E}^{ij}{}_k(\mathbf{x})L^k \\ \qquad + \dfrac{m}{2}\left(g_{ab}\partial_n g^{ab}(\mathbf{x})\right) \\ \qquad \times\left(\mathrm{E}^i{}_{kl}\mathrm{E}^j{}_m{}^n - \mathrm{E}^j{}_{kl}\mathrm{E}^i{}_m{}^n\right)x^m x^k\dot{x}^l \\ \qquad + q\mathrm{E}^i{}_{kl}(\mathbf{x})\mathrm{E}^j{}_{mn}(\mathbf{x})x^k x^m\mathcal{F}^{ln}(\mathbf{X}). \end{cases} \tag{46}$$

In order to restore the sO(3) symmetry as in the flat case we perform the following transformation on the angular momentum:

$$L^i(\mathbf{X}) \to \mathcal{L}^i(\mathbf{X}) = L^i(\mathbf{X}) + M^i(\mathbf{X}), \tag{47}$$

and we thus impose the constraints

$$\begin{cases} \left[x^i, \mathcal{L}^j\right] = \mathrm{E}^{ij}{}_k x^k, \\ \left[\dot{x}^i, \mathcal{L}^j\right] = \mathrm{E}^{ij}{}_k \dot{x}^k, \\ \left[\mathcal{L}^i, \mathcal{L}^j\right] = \mathrm{E}^{ij}{}_k \mathcal{L}^k. \end{cases} \tag{48}$$

The first equation in (48) implies that the new angular momentum is velocity independent, $M^i(\mathbf{X}) = M^i(\mathbf{x})$; the second equation in (48) then gives

$$\left[\dot{x}^i, M^j\right] = -\frac{q}{m}\mathrm{E}^j{}_{lm} F^{im} x^l. \tag{49}$$

This last equation is similar to (31) found for the flat space case, and the angular momentum M is then the Poincaré momentum

$$M^i(\mathbf{x}) = \frac{1}{2} q \mathrm{E}_{jkl}(\mathbf{x}) F^{jl}(\mathbf{x}) x^i x^k = -q\left(\mathbf{x}\cdot\mathbf{B}\right) x^i. \tag{50}$$

Still, this kind of relation, (50), implies a Dirac magnetic monopole field

$$\mathbf{B} = \frac{g}{4\pi}\frac{\mathbf{x}}{\|\mathbf{x}\|^3}. \tag{51}$$

We have then shown that the sO(3) symmetry algebra in curved space is restored by introducing the same Dirac magnetic monopole as we introduce in the flat space case.

4 Lorentz algebra in a curved space

The natural extension of the previous computation is obviously the study of the Lorentz algebra in a curved space. In this section we consider the following Hamiltonian:

$$H = \frac{1}{2} m g_{\mu\upsilon}(\mathbf{x}) \dot{x}^\mu \dot{x}^\nu, \tag{52}$$

where $g_{\mu\nu}(\mathbf{x})$ is the metric of the Riemannian manifold $\mathcal{M}$.

4.1 Maxwell equations

The simple commutation relations between positions and velocities coordinates are

$$\begin{cases} \left[x^\mu, x^\nu\right] = 0, \\ \left[x^\mu, \dot{x}^\nu\right] = \dfrac{1}{m} g^{\mu\nu}(\mathbf{x}), \\ \left[\dot{x}^\mu, \dot{x}^\nu\right] = \dfrac{q}{m^2}\mathcal{F}^{\mu\nu}(\mathbf{X}), \\ \left[\dot{x}_\mu, \dot{x}_\nu\right] = \dfrac{q}{m^2} F_{\mu\nu}(\mathbf{x}) \\ \qquad = \dfrac{q}{m^2}\mathcal{F}_{\mu\nu}(\mathbf{X}) + \left(\partial_\nu g_{\mu\rho} - \partial_\mu g_{\nu\rho}\right)\dot{x}^\rho, \end{cases} \tag{53}$$

from which we easily derive the equation of motion:

$$m\ddot{x}^\mu = -\Gamma^{\mu\nu\rho}\dot{x}_\nu \dot{x}_\rho + q F^{\mu\nu}(\mathbf{x})\dot{x}_\nu. \tag{54}$$

The Jacobi identity $J\left(\dot{x}_\mu, \dot{x}_\nu, \dot{x}_\rho\right) = 0$ directly gives

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0, \tag{55}$$

which is the first group of the Maxwell equations. The other Jacobi identity $J\left(\dot{x}_\mu, \dot{x}^\nu, \dot{x}_\nu\right) = 0$ gives us the relation

$$\left[\dot{x}_\nu, \left[\dot{x}_\mu, \dot{x}^\nu\right]\right] = -\left[\dot{x}^\nu, \left[\dot{x}_\nu, \dot{x}_\mu\right]\right] = \frac{q}{m^3}\,\partial^\nu F_{\nu\mu}, \tag{56}$$

where the quantity $\partial^\nu F_{\nu\mu}$ is a priori different from zero. We then define the second group of the generalized Maxwell equations by

$$\begin{Bmatrix} \partial^\nu F_{\nu\mu} = 0 \text{ , for the vacuum,} \\ \partial^\nu F_{\nu\mu} = j_\mu \text{ , for a medium with a current density } j^\alpha. \end{Bmatrix} \tag{57}$$

4.2 Lorentz algebra and Poincaré momentum

It is convenient to define the angular quadrimomentum by

$$L_{\mu\nu} = m\left(x_\mu \dot{x}_\nu - x_\nu \dot{x}_\mu\right), \tag{58}$$

which gives a deformed Lorentz algebra with the following structure:

$$\begin{cases} \left[x_\mu, L_{\rho\sigma}\right] = \left\{x_\mu, L_{\rho\sigma}\right\} \\ \quad = g_{\mu\sigma}(\mathbf{x})x_\rho - g_{\mu\rho}(\mathbf{x})x_\sigma + x_\rho x^\lambda \partial_\sigma g_{\mu\lambda}(\mathbf{x}) \\ \quad - x_\sigma x^\lambda \partial_\rho g_{\mu\lambda}(\mathbf{x}), \\ \left[\dot{x}_\mu, L_{\rho\sigma}\right] = \left\{\dot{x_\mu}, L_{\rho\sigma}\right\} + \dfrac{q}{m}\left(F_{\mu\sigma}(\mathbf{x})\dot{x_\rho} - F_{\mu\rho}(\mathbf{x})\dot{x_\sigma}\right) \\ \quad = g_{\mu\sigma}(\mathbf{x})\dot{x_\rho} - g_{\mu\rho}(\mathbf{x})\dot{x_\sigma} + \dot{x}_\rho x^\lambda \partial_\sigma g_{\mu\lambda}(\mathbf{x}) \\ \quad - \dot{x}_\sigma x^\lambda \partial_\rho g_{\mu\lambda}(\mathbf{x}) + \dfrac{q}{m}\left(F_{\mu\sigma}(\mathbf{x})\dot{x_\rho} - F_{\mu\rho}(\mathbf{x})\dot{x_\sigma}\right), \\ \left[L_{\mu\nu}, L_{\rho\sigma}\right] = \left\{L_{\mu\nu}, L_{\rho\sigma}\right\} \\ \quad + q(x_\mu x_\rho F_{\nu\sigma}(\mathbf{x}) - x_\nu x_\rho F_{\mu\sigma}(\mathbf{x}) + x_\mu x_\sigma F_{\rho\nu}(\mathbf{x}) \\ \qquad - x_\nu x_\sigma F_{\rho\mu}(\mathbf{x})) \\ \quad = g_{\mu\rho}(\mathbf{x})L_{\nu\sigma} - g_{\nu\rho}(\mathbf{x})L_{\mu\sigma} + g_{\mu\sigma}(\mathbf{x})L_{\rho\nu} - g_{\nu\sigma}(\mathbf{x})L_{\rho\mu} \\ \quad + m\left(x_\rho \dot{x}_\nu x^\lambda \partial_\mu g_{\lambda\sigma}(\mathbf{x}) - x_\nu \dot{x}_\rho x^\lambda \partial_\sigma g_{\lambda\mu}(\mathbf{x})\right. \\ \quad + x_\mu \dot{x}_\rho x^\lambda \partial_\sigma g_{\lambda\mu}(\mathbf{x}) - x_\rho \dot{x}_\mu x^\lambda \partial_\mu g_{\lambda\sigma}(\mathbf{x}) \\ \quad + x_\nu \dot{x}_\sigma x^\lambda \partial_\mu g_{\lambda\rho}(\mathbf{x}) - x_\sigma \dot{x}_\nu x^\lambda \partial_\rho g_{\lambda\mu}(\mathbf{x}) \\ \quad \left. + x_\sigma \dot{x}_\mu x^\lambda \partial_\rho g_{\lambda\nu}(\mathbf{x}) - x_\mu \dot{x}_\sigma x^\lambda \partial_\nu g_{\lambda\rho}(\mathbf{x})\right) \\ \quad + q(x_\mu x_\rho F_{\nu\sigma}(\mathbf{x}) - x_\nu x_\rho F_{\mu\sigma}(\mathbf{x}) + x_\mu x_\sigma F_{\rho\nu}(\mathbf{x}) \\ \qquad - x_\nu x_\sigma F_{\rho\mu}(\mathbf{x})). \end{cases} \tag{59}$$

We apply here the same scheme as used for the sO(3) algebra; we restore the Lorentz symmetry by using the following angular quadrimomentum transformation law:

$$L_{\mu\nu}(\mathbf{X}) \to \mathcal{L}_{\mu\nu}(\mathbf{X}) = L_{\mu\nu}(\mathbf{X}) + M_{\mu\nu}(\mathbf{X}) \tag{60}$$

and by requiring the usual structure

$$\begin{cases} [x_\mu, \mathcal{L}_{\rho\sigma}] &= \{x_\mu, \mathcal{L}_{\rho\sigma}\} &= g_{\mu\sigma}x_\rho - g_{\mu\rho}x_\sigma, \\ [\dot{x}_\mu, \mathcal{L}_{\rho\sigma}] &= \{\dot{x}_\mu, \mathcal{L}_{\rho\sigma}\} &= g_{\mu\sigma}\dot{x}_\rho - g_{\mu\rho}\dot{x}_\sigma, \\ [\mathcal{L}_{\mu\nu}, \mathcal{L}_{\rho\sigma}] &= \{\mathcal{L}_{\mu\nu}, \mathcal{L}_{\rho\sigma}\} &= g_{\mu\rho}\mathcal{L}_{\upsilon\sigma} - g_{\nu\rho}\mathcal{L}_{\mu\sigma} \\ & & \quad + g_{\mu\sigma}\mathcal{L}_{\rho\nu} - g_{\nu\sigma}\mathcal{L}_{\rho\mu}. \end{cases} \tag{61}$$

From (61) we easily deduce that the quadrimomentum $M_{\mu\nu}$ is only position dependent, $M_{\mu\nu}(\mathbf{X}) = M_{\mu\nu}(\mathbf{x})$. Then (61) also gives

$$[\dot{x}_\mu, M_{\rho\sigma}] = \frac{q}{m}(F_{\mu\sigma}x_\rho - F_{\mu\rho}x_\sigma). \tag{62}$$

This result (62) with the third relation given in (61) give us the following relation:

$$\begin{aligned} & g_{\mu\rho}M_{\nu\sigma} - g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\rho\nu} - g_{\nu\sigma}M_{\rho\mu} \\ & = q(F_{\nu\sigma}x_\mu x_\rho - F_{\mu\sigma}x_\nu x_\rho + F_{\rho\nu}x_\mu x_\sigma - F_{\rho\mu}x_\nu x_\sigma), \end{aligned} \tag{63}$$

which will define the quadrimomentum $M_{\mu\nu}$.

First, let us consider the case $\nu = \sigma = i$, where $i = 1, 2, 3$, and with a sum over i. Equation (63) then becomes

$$\begin{aligned} & -g^i{}_\rho M_{\mu i}(\mathbf{x}) + g_\mu{}^i M_{\rho i}(\mathbf{x}) - 3M_{\rho\mu}(\mathbf{x}) \\ & = q(-F_{\mu i}(\mathbf{x})x^i x_\rho + F_{\rho i}(\mathbf{x})x_\mu x^i - F_{\rho\mu}(\mathbf{x})\mathbf{x}^2). \end{aligned} \tag{64}$$

Now setting $\rho = j$ and $\mu = k$, we obtain

$$M_{ij} = q(F_{ij}x^k x_k - F_{jk}x^k x_i - F_{ki}x^k x_j), \tag{65}$$

which is nothing more than the generalization of the previously found equations (32) and (50). M_{ij} is then the quadrimomentum related to the previously found Poincaré momentum M_i. Indeed using the definition of the quadrimomentum $M_i = \varepsilon_i{}^{jk}M_{jk}$, we retrieve for the spatial degrees of freedom ($i = 1, 2, 3$) the Poincaré momentum

$$\mathbf{M} = -q(\mathbf{x}\cdot\mathbf{B})\mathbf{x}. \tag{66}$$

Using now (66) and (62) we obtain the set of equations

$$\begin{cases} x_i B_j + x_j B_i = -x_j x^k \partial_i B_k, \\ F_{0j}x_i - F_{0i}x_j = (\mathbf{x}\times\mathbf{E})_{k\neq i,j} = 0, \end{cases} \tag{67}$$

whose solutions are radial vector fields centered at the origin

$$\begin{cases} \mathbf{B} = \dfrac{g}{4\pi}\dfrac{\mathbf{x}}{\|\mathbf{x}\|^3}, \\ \mathbf{E} = q' f(\mathbf{x})\mathbf{x}. \end{cases} \tag{68}$$

It is straightforward to see that our results are still valid for a flat quadridimensional space. We have then shown that the Lorentz symmetry in a curved and in a flat space is restored if the magnetic field is the Dirac monopole magnetic field and if the electric field is radial.

Consider now the "boost" part of (63). For $\rho = 0$, and $\mu = j$, (64) corresponds to the temporal components of the Poincaré tensor,

$$M_{0j} = q(-F_{ji}x^i x_0 - F_{0i}x_j x^i + F_{0j}\mathbf{x}^2). \tag{69}$$

This relation can also be written

$$M_{0j} = q\left[-(\mathbf{x}\times\mathbf{B})_j x_0 - (\mathbf{x}\cdot\mathbf{E})x_j + \mathbf{x}^2 E_j\right], \tag{70}$$

which for the solution (68) gives the result

$$M_{0j} = 0.$$

The temporal component of the generalized angular momentum $M_{\mu\nu}$ which restores the Lorentz symmetry mixes the electric and the magnetic fields in such way that it is equal to zero, whereas the spatial components of $M_{\mu\nu}$ are only magnetic field dependent and correspond to the usual Poincaré momentum components. It is important to be precise in that the equations (68) imply that the source of the electromagnetic field is created by a Schwinger dyon of magnetic charge g and electric charge q'.

In the next section we extend our model by adding to the electromagnetic $F^{\mu\nu}$ tensor its dual ${}^*F^{\mu\nu}$ in a curved space and we deduce in the frame of gravitoelectromagnetism a quantization of the dyon's mass.

4.3 Gravitoelectromagnetism

In order to interpret the experimental tests of gravitation theories, the parametrized post-Newtonian formalism (PPN) is often used [25], where the limit of low velocities and small stresses is taken. In this formalism, gravity is described by a general type metric containing dimensionless constants call PPN-parameters, which are powerful tools in theoretical astrophysics. This formalism was applied by Braginski et al. [26] to propose laboratory experiments to test relativistic gravity and in particular to study gravitoelectromagnetism. They analyzed magnetic and electric type gravity using a truncated and rewritten version of the PPN formalism by deleting certain parameters not present in general relativity and all gravitational non-linearities. In a theoretical paper [9] Mashhoon has considered several important quantities relative to this theory, like field equations, gravitational Larmor precession or the stress-energy tensor. He introduced gravitoelectromagnetism which is based upon the formal analogy between gravitational Newton potential and electric Coulomb potential. A long time ago, Holzmuller [27] and Tisserand [28] have already postulated gravitational electromagnetic components for the gravitational influence of the sun on the motion of planets. More recently, Mashhoon [9] has considered a particle of inertial mass m which has also a gravitoelectric charge $q_E = -m$ and gravitomagnetic charge $q_M = -2m$, the numerical factor 2 coming from the spin character of the gravitational field. In the final part of this work we apply our formalism to this last idea.

Suppose that gravitation creates a gravitoelectromagnetic field characterized by $F_{\mu\nu}(\mathbf{x})$ and ${}^*F_{\mu\nu}(\mathbf{x})$, where the symbol * stands here for the Hodge duality. We then have

$$[\dot{x}_\mu, \dot{x}_\nu] = -\frac{1}{m^2}(qF_{\mu\nu}(\mathbf{x}) + g\,{}^*F_{\mu\nu}(\mathbf{x})), \tag{71}$$

where q and g are respectively the gravitoelectric and the gravitomagnetic charge of a Schwinger dyon moving in this gravitoelectromagnetic field. At the end of this section we will choose like Mashoon an explicit relation between the charges q and g and the inertial mass m. By a direct application of our formalism developed in Sect. 4 the equation of motion of our dyon particle is obtained:

$$m\ddot{x}^{\mu} = -\Gamma^{\mu\nu\rho}\dot{x}_{\nu}\dot{x}_{\rho} + \left(qF^{\mu\nu}(\mathbf{x}) + g^{*}F_{\mu\nu}(\mathbf{x})\right)\dot{x}_{\nu}. \quad (72)$$

In order to restore the Lorentz symmetry in a curved space we introduce the generalized angular momentum now expressed in terms of the electromagnetic field and its dual. For the spatial components we have the following equation:

$$\begin{aligned} M_{ij} = {} & q(F_{ij}x^k x_k - F_{jk}x^k x_i - F_{ki}x^k x_j) \\ & + g({}^{*}F_{ij}x^k x_k - {}^{*}F_{jk}x^k x_i - {}^{*}F_{ki}x^k x_j), \end{aligned} \quad (73)$$

which shows that the new angular momentum is the sum of two contributions, a gravitomagnetic one and a gravitoelectric one:

$$\mathbf{M} = \mathbf{M_m} + \mathbf{M_e}, \quad (74)$$

where

$$\begin{cases} \mathbf{M_m} = -q(\mathbf{x}\cdot\mathbf{B})\mathbf{x}, \\ \mathbf{M_e} = \quad g(\mathbf{x}\cdot\mathbf{E})\mathbf{x} \end{cases} \quad (75)$$

are respectively the gravitomagnetic and gravitoelectric angular momenta. Introducing the notation $\mathbf{P} = q'\mathbf{B} - g'\mathbf{E}$ we can show that the Lorentz symmetry is restored by the Poincaré-like angular momentum

$$\mathbf{M} = -(\mathbf{x}\cdot\mathbf{P})\mathbf{x}, \quad (76)$$

where $\mathbf{P}$ has the Dirac-like form

$$\mathbf{P} \sim \frac{\mathbf{x}}{4\pi\,\|\mathbf{x}\|^3}. \quad (77)$$

A possible choice for the electromagnetic field is then to choose a dyon source responsible for the Dirac and the Coulomb monopole fields,

$$\begin{cases} \mathbf{B} = \dfrac{g'}{4\pi}\dfrac{\mathbf{x}}{\|\mathbf{x}\|}, \\ \mathbf{E} = -\dfrac{q'}{4\pi}\dfrac{\mathbf{x}}{\|\mathbf{x}\|}, \end{cases} \quad (78)$$

so that

$$\mathbf{P} = (q'g + g'q)\,\frac{\mathbf{x}}{4\pi\,\|\mathbf{x}\|^3}. \quad (79)$$

Consequently we have a gravitoelectromagnetic dyon (characterized by its mass m and its charges q and g) moving in a gravitoelectromagnetic monopole field created by a dyon (characterized by its mass m' and its charges q' and g').

From the quantization of (76) with P given by (79) we deduce the following relation:

$$\frac{q'g + g'q}{4\pi} = \frac{n\hbar}{2}. \quad (80)$$

We postulate as in [9] the following relations between the gravitoelectromagnetic charges and the inertial masses:

$$\begin{cases} q \;= a\sqrt{G}m, \\ g \;= b\sqrt{G}m, \\ q' = a\sqrt{G}m', \\ g' = b\sqrt{G}m', \end{cases} \quad (81)$$

where a and b are two constants and G is the gravitational constant. We also have

$$\frac{q_M}{q_E} = \frac{q'_M}{q'_E} = \frac{b}{a} = s,$$

which is Mashhoon's relation between electric and magnetic charges and the spin of the gauge boson interaction. In the gravitoelectromagnetic theory we naturally choose $s = 2$; we then deduce the following mass condition for the dyons:

$$m\,m' = nA\frac{hc}{G} = nAM_{\mathrm{P}}^2, \quad (82)$$

where A is a dimensionless constant and n is an integer number in the Schwinger formalism (bosonic spectrum) and a half integer number in the Dirac formalism (fermionic and bosonic spectrum), M_{P} being the Planck mass.

5 Conclusion

In this paper we have introduced a Poisson structure with dynamics defined through a covariant Hamiltonian. Our formalism could also be expressed in terms of a generalization of the Moyal brackets defined on the tangent bundle space. In non-commutative theories the parameter Θ^{ij} expresses the non-commutativity of the positions, whereas in our construction it is the electromagnetic field $F^{ij}(x)$ which induces the non-commutativity of the velocities. Our aim was to find the generalized angular momentum which enables us to restore the Lie algebra symmetry (sO(3) and Lorentz) of the angular momentum which is broken by the electromagnetic field, i.e. by the non-commutativity of the velocities. The solution is the Poincaré angular momentum in the flat space case as well as in the curved space case. The formalism was applied in the framework of the gravitoelectromagnetism, where it was shown that the Dirac and the Coulomb monopoles allow one to build both a magnetic and an electric Poincaré-like angular momentum, which restores the Lorentz algebra symmetry in a curved space. The quantization of the total angular momentum and Mashoon's relation between the gravitoelectromagnetic charges and the inertial masses lead to a qualitative condition on the mass spectrum.

It would be interesting to extend our approach to the context of the non-commutativity theory where $\Theta^{ij} \neq 0$; work in that direction is in progress [3].

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